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Approximate Reasoning Based on Similarity

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Abstract. The connection between similarity logic and the theory of closure operators is examined. Indeed one proves that the consequence relation defined in [14] can be obtained by composing two closure operators and that the resulting operator is still a closure operator. Also, we extend any similarity into a similarity which is compatible with the logical equivalence, and we prove that this gives the same consequence relation.

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1 Introduction

Due to the essential vagueness and approximation of human thinking, the logical treatment of uncertainty is of increasing importance in artificial intelligence and related research. Nowadays, a considerable number of logical systems have been carried out as formalizations of vague concepts and approximate reasoning (for example, see Gargov [4], Gerla [5], Goguen [6], Hajek [7], Lano [8], Marquis [9], Novak [10], Pavelka [11] and Takeuti and Titani [12]). However, reasoning in these logics is still exact, i. e., in order to apply an inference rule, the antecedent clauses of this rule must be equal either to some premises or to logical axioms or previously proven formulas.

Recently, Ying [13, 14] proposed a new approach in which we can really make approximate reasonings, i. e., it is possible to allow the antecedent clauses of a rule to match its premises (or logical axioms or previously proven formulas) only approximately. The starting point is a similarity R defined in the set of propositional variables and its "natural" extension $\overline{R}$ to the whole set of propositional formulas. Subsequently, Biacino and Gerla [2] generalized the definition of an approximate consequence operator given in [14] and clarified its connection with Pavelka's logic.

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This paper is a continuation of [14] and [2] and considers the following problems:

(1) Is the similarity-based consequence operator we propose a closure operator ?

(2) is it compatible with the logical equivalence ?

Again, we observe that the natural extension $\overline{R}$ of R is not compatible with the logical equivalence $\equiv$. So we can define the least similarity $\overline{R}_{\mathrm{e}}$ containing $\overline{R}$ and compatible with $\equiv$.

(3) What about the relationship between the similarity logics based on $\overline{R}_e$ and on $\overline{R}$, respectively ?

2 Preliminaries

In what follows we denote by L a complete and infinitely distributive lattice. Also, if S is a nonempty set, by L^S we denote the class of the L-subsets of S, that is the maps $s : S \longrightarrow L$. The *inclusion* relation between two fuzzy subsets s_1 and s_2 is defined by setting

$$s_1 \subseteq s_2 \quad \text{iff} \quad s_1(x) \leq s_2(x) \text{ for every } x \in S.$$

The *intersection* and *union* operations are defined by setting, for any $x \in S$,

$$(s_1 \cap s_2)(x) = s_1(x) \wedge s_2(x) \quad \text{and} \quad (s_1 \cup s_2)(x) = s_1(x) \vee s_2(x).$$

In a similar way we define the infinitary unions and intersections. Given an L-subset s and $\lambda \in L$, we call *closed* λ-*cut* the subset

$$C(s, \lambda) = \{x \in L : s(x) \geq \lambda\}.$$

A *fuzzy closure operator* on a set S is a map $J : L^S \longrightarrow L^S$ such that, for every s, s_1, s_2 in L^S,

(a) $J(s) \supseteq s$ (inclusion);

(b) If $s_1 \subseteq s_2$, then $J(s_1) \subseteq J(s_2)$ (monotonicity);

(c) $J(J(s)) = J(s)$ (idempotence).

Moreover, we say that s is *closed with respect to* J provided that s is a fixed point of J, i. e., $J(s) = s$. In account of the inclusion property, s is closed with respect to J iff $J(s) \subseteq s$. If J_1 and J_2 are fuzzy closure operators then the product $J_1 \circ J_2$ satisfies the inclusion property and the monotonicity but is not idempotent, in general. Nevertheless the following rather obvious propositions hold.

Proposition 2.1. *Let J_1 and J_2 be fuzzy closure operators. Then, the following are equivalent:*

(i) *$J_1 \circ J_2$ is a fuzzy closure operator.*

(ii) *$J_1 \circ J_2 \circ J_1 = J_1 \circ J_2$.*

(iii) *$J_2 \circ J_1 \circ J_2 = J_1 \circ J_2$.*

(iv) *$J_1 \circ J_2(s)$ is closed with respect to J_2 for any fuzzy subset s.*

Proof.

(i) ⇒ (ii). Since $J_1(s) \supseteq s$, we have that $J_1(J_2(J_1(s))) \supseteq J_1(J_2(s))$. Conversely, since $J_2(s) \supseteq s$, we have $J_1 J_2(J_1(J_2(s)))) \supseteq J_1(J_2(J_1(s))$ and therefore $J_1(J_2(s)) = J_1(J_2(J_1(J_2(s)))) \supseteq J_1(J_2(J_1(s)))$.

(ii) $\Rightarrow$ (i). From $J_1 \circ J_2 \circ J_1 = J_1 \circ J_2$ it follows that $J_1 \circ J_2 \circ J_1 \circ J_2 = J_1 \circ J_2 \circ J_2 = J_1 \circ J_2$.

(i) $\Rightarrow$ (iii). Since J_2 satisfies the inclusion property, $J_2(J_1(J_2(s))) \supseteq J_1(J_2(s))$. Moreover, by the idempotence of $J_1 \circ J_2$ and the inclusion property of J_1, $J_1(J_2(s)) = J_1(J_2(J_1(J_2(s)))) \supseteq J_2(J_1(J_2(s)))$.

(iii) $\Rightarrow$ (i). From $J_2 \circ J_1 \circ J_2 = J_1 \circ J_2$ it follows that $J_1 \circ J_2 \circ J_1 \circ J_2 = J_1 \circ J_1 \circ J_2 = J_1 \circ J_2$.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are immediate. □

Notice that $J_1 \circ J_2$ fuzzy closure operator does not entail that $J_2 \circ J_1$ is a fuzzy closure operator, in general. As an example, let J_1 be the usual closure operator associated with a fuzzy topology and define J_2 by setting $J_2(s) = s \cup v$, where v is any open (and not closed) fuzzy subset. Then $J_1(J_2(s)) = \overline{s \cup v}$, and therefore $J_2 J_1(J_2(s)) = J_1(J_2(s))$. This proves that $J_1 \circ J_2$ is a closure operator. Instead, since $J_1(J_2(J_1(s))) = \overline{s} \cup \overline{v} \neq \overline{s} \cup v = J_2(J_1(s))$, $J_2 \circ J_1$ is not a closure operator.

Proposition 2.2. *Let J_1 and J_2 be fuzzy closure operators. Then the following are equivalent:*

(i) *Both $J_1 \circ J_2$ and $J_2 \circ J_1$ are fuzzy closure operators.*

(ii) $J_1 \circ J_2 = J_2 \circ J_1$.

Proof. From Proposition 2.1 it follows that if both $J_1 \circ J_2$ and $J_2 \circ J_1$ are fuzzy closure operators, then $J_2 \circ J_1 = J_1 \circ J_2 \circ J_1 = J_1 \circ J_2$. The converse implication is immediate. □

3 Similarities

An *L-valued binary relation on S* is an L-subset $R : S \times S \longrightarrow L$ of the set $S \times S$. Moreover

- if $R(x,x) = 1$ for any $x \in S$, then R is said to be *reflexive*;
- if $R(x,y) = R(y,x)$ for any $x, y \in S$, then R is said to be *symmetric*;
- if $R(x,y) \wedge R(y,z) \leq R(x,z)$ for any $x, y, z \in S$, then R is said to be *transitive*.

A reflexive, symmetric and transitive L-valued binary relation on S is called a *similarity on S*. In the following we denote by R_λ the λ-cut $C(R,\lambda)$ of R. It is easy to prove that R is a similarity iff every λ-cut $R_\lambda = \{(x,y) : R(x,y) \geq \lambda\}$ is a (classical) equivalence relation in S. Any similarity relation R is associated with an interesting fuzzy closure operator. Indeed, we define J_R by setting

$$J_R(s)(x) = \bigvee_{y \in S} (s(y) \wedge R(x,y)).$$

In a sense, $J_R(s)$ is the L-subset of elements that are similar to some element in s. Obviously, s is a fixed point of J_R iff for every $x, y \in S$, $s(x) \geq s(y) \wedge R(x,y)$. In this case we say that s is *closed with respect to R*, too.

As it is known, given any L-relation R we can build up the similarity relation generated by R. To illustrate such a possibility, we have to give some definitions.

Definition 3.1. Let R and $\overline{R}$ be L-valued binary relations on a set S. Then $\overline{R}$ is called the *reflexive* (or *symmetric*, or *transitive*) *closure* of R if

· $\overline{R}$ is reflexive (or symmetric, or transitive, respectively),

· $\overline{R} \supseteq R$,

· $T \supseteq \overline{R}$ provided that T is an L-valued binary relation such that $T \supseteq R$ and T is reflexive (or symmetric, or transitive, respectively).

The reflexive, symmetric and transitive closures of R are denoted by $r(R)$, $s(R)$, and $t(R)$, respectively.

In the following if R_1 and R_2 are binary L-relations on S, then $R \circ S$ is the L-relation on S defined by setting for any x, $y \in S$,

$$(R_1 \circ R_2)(x,y) = \bigvee_{z \in Y} (R_1(x,z) \wedge R_2(z,y)).$$

The powers R^n are defined by induction on n by setting, $R^1 = R$ and $R^{n+1} = R \circ R^n$. We denote by Δ the diagonal of S, i. e., for any x, $y \in S$,

$$\Delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Also, we denote by R^{-1} the inverse of R, i. e., for any x, $y \in Y$, $R^{-1}(x,y) = R(y,x)$.

The proof of the following proposition is immediate.

Proposition 3.1. *Given an L-relation R, then $r(R) = \Delta \cup R$, $s(R) = R \cup R^{-1}$, and $t(R) = \bigcup_{n \in \mathbb{N}} R^n$.*

The following proposition, whose proof is immediate, enables us to obtain the similarity generated by a given L-relation.

Proposition 3.2. *Define the operator* $\mathrm{Sim} : L^{S\times S} \longrightarrow L^{S\times S}$ *by setting for any R in $L^{S\times S}$,* $\mathrm{Sim}(R) = t(r(s(R)))$. *Then* Sim *is a fuzzy closure operator and* $\mathrm{Sim}(R)$ *is the smallest similarity containing R.*

4 Approximate logical consequences

Let $T = \{F, \Rightarrow\}$, where F is a constant (called *false*) and $\Rightarrow$ a binary connective, and let X be a set whose elements we call *propositional variables*. Then we denote by $P(X)$ the language of the propositional calculus on X, i. e., the free T-algebra on X (cf. e. g. Barnes and Mack [3, p. 12]), and by $\mathrm{Taut} \subseteq P(X)$ the set of all (classical) tautologies. For any $p \in P(X)$, let $[p]$ be the equivalence class of p under the logical equivalence $\equiv$, i. e., $[p] = \{q \in P(X) : q \equiv p\}$. Also, for any $A \subseteq P(X)$, let $[A] = \bigcup_{p \in A}[p]$. If $A \subseteq P(X)$ we say that A *is closed under* $\equiv$ provided $p \equiv q$ implies that $p \in A$ iff $q \in A$. It is immediate that $[A]$ is the least set of formulas closed with respect to $\equiv$ and containing A. Therefore A is closed with respect to $\equiv$ iff $A = [A]$. We can extend such notions to L-subsets as follows. An L-subset s is said to be *closed under* $\equiv$ if, for any p, $q \in P(X)$, $p \equiv q$ entails $s(p) = s(q)$. Also, we set $[s](p) = \mathrm{Sup}\{s(q) : q \equiv p\}$. Then $[s]$ is the least L-subset closed with respect to $\equiv$ and containing s. Obviously, s is closed under $\equiv$ if and only if $s = [s]$.

Definition 4.1. Let R be an L-valued binary relation on X. The *natural extension* $\overline{R}$ on $P(X)$ of R is given by induction as follows:

· $\overline{R}(p,q) = R(p,q)$ if p, $q \in X$;

· $\overline{R}(F,q) = \overline{R}(q,F) = 0$ if $q \neq F$, and $\overline{R}(F,F) = 1$;

· $\overline{R}(p,q) = \overline{R}(x,x') \wedge \overline{R}(y,y')$ if $p = (x \Rightarrow y)$ and $q = (x' \Rightarrow y')$;

· $\overline{R}(p,q) = 0$ otherwise.

Clearly, if R is reflexive, symmetric or transitive, so is $\overline{R}$. In particular, if R is a similarity, then $\overline{R}$ is a similarity, too. Another way to define $\overline{R}$ is the following. For any $\alpha \in P(X)$ let $\mathrm{Var}(\alpha)$ denote the set of propositional variables in α. *Schemes* are particular formulas defined by assuming that every propositional variable is a schema, and if α and β are schemes and $\mathrm{Var}(\alpha) \cap \mathrm{Var}(\beta) = \emptyset$, then $(\alpha \Rightarrow \beta)$ is also a schema. We denote by $\alpha(y_1, \ldots, y_n)$ a schema α such that $\mathrm{Var}(\alpha) = \{y_1, \ldots, y_n\}$. A *substitution of propositional variables in* α is a map from $\{y_1, \ldots, y_n\}$ into X. We denote by $(x_1/y_1, \ldots, x_n/y_n)$ a substitution where x_i is the image of y_i $(i = 1, 2, \ldots, n)$, and we write $p = \alpha(x_1/y_1, \ldots, x_n/y_n)$ (in brief $p = \alpha(x_1, \ldots, x_n)$) to denote the formula obtained by substituting y_i with x_i. We say that p and p' have *the same structure* if there exists a schema $\alpha(y_1, \ldots, y_n)$ such that $p = \alpha(x_1/y_1, \ldots, x_n/y_n)$ and $p' = \alpha(x'_1/y_1, \ldots, x'_n/y_n)$. In this case $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ are called the *characteristics* of p and p', respectively.

Proposition 4.1. *For any p, $p' \in P(X)$ we have*

- $\overline{R}(p, p') = 0$ *if p and p' have not the same structure;*
- $\overline{R}(p, p') = \bigwedge_{i=1}^{n} R(x_i, x'_i)$ *if p and p' have the same structure and $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ are the characteristics of p and p', respectively.*

Proof. By induction on the length $\mathrm{l}(p)$ of p. □

The following definition is on the basis of similarity logic.

Definition 4.2. For any $s \in L^{P(X)}$ and $B \subseteq P(X)$ let

$$\widetilde{R}(s, B) = \bigwedge_{q \in B} \bigvee_{p \in P(X)} (s(p) \wedge \overline{R}(p, q)).$$

In a sense $\widetilde{R}(s, B)$ is the truth degree of the claim "every element in B is equivalent to an element in s". It can be seen as a generalization of the inclusion relation. Indeed, if s is the crisp set A and R is the identity relation, then $\widetilde{R}(A, B) = 1$ iff B is enclosed in A. It is immediate that $\widetilde{R}(s, \emptyset) = 1$ and that $\widetilde{R}(s, B) = \bigwedge_{q \in B} J_{\overline{R}}(s)(q)$. As a consequence, if s is closed with respect to $\overline{R}$, then $\widetilde{R}(s, B) = \bigwedge_{q \in B} s(q)$.

The following proposition shows that $\widetilde{R}$ is transitive, in a sense.

Proposition 4.2. *For any $s \in L^{P(X)}$ and A, $B \subseteq P(X)$,*

$$\widetilde{R}(s, A) \wedge \widetilde{R}(A, B) \leq \widetilde{R}(s, B).$$

Proof. From Definition 4.2 we have

$$\begin{aligned}
\widetilde{R}(s, A) \wedge \widetilde{R}(A, B) &= \widetilde{R}(s, A) \wedge \left(\bigwedge_{r \in B} \bigvee_{q \in A} \overline{R}(q, r)\right) \\
&= \bigwedge_{r \in B} \left(\widetilde{R}(s, A) \wedge \bigvee_{q \in A} \overline{R}(q, r)\right) \\
&= \bigwedge_{r \in B} \bigvee_{q \in A} \left(\widetilde{R}(s, A) \wedge \overline{R}(q, r)\right) \\
&= \bigwedge_{r \in B} \bigvee_{q \in A} \left[\bigwedge_{q' \in A} \bigvee_{p \in P(X)} (s(p) \wedge \overline{R}(p, q')) \wedge \overline{R}(q, r)\right] \\
&\leq \bigwedge_{r \in B} \bigvee_{q \in A} \left[\bigvee_{p \in P(X)} (s(p) \wedge \overline{R}(p, q)) \wedge \overline{R}(q, r)\right] \\
&= \bigwedge_{r \in B} \bigvee_{q \in A} \bigvee_{p \in P(X)} (s(p) \wedge \overline{R}(p, q) \wedge \overline{R}(q, r)) \\
&\leq \bigwedge_{r \in B} \bigvee_{p \in P(X)} (s(p) \wedge \overline{R}(p, r)) \\
&= \widetilde{R}(s, B).
\end{aligned}$$

□

Now we are able to give the main definition in similarity logic.

Definition 4.3. Let R be an L-valued binary relation on X. For any $s \in L^{P(X)}$ and $q \in P(X)$, *the degree to which q follows from s* is given by

$$\mathrm{Con}_R(s,q) = \bigvee\{\widetilde{R}(\mathrm{Taut} \cup s, B) : B \vdash q\},$$

where $B \vdash q$ means that q is a consequence of B in (classical) propositional logic.

Notice that if q is a tautology, then $\mathrm{Con}_R(s,q) = \widetilde{R}(\mathrm{Taut} \cup s, \emptyset) = 1$. Also, in defining $\mathrm{Con}_R(s,q)$ we can refer only to finite sets B of formulas. Finally observe that if s is closed with respect to $\overline{R}$ and $s \supseteq \mathrm{Taut}$, then

$$\mathrm{Con}_R(s,q) = \bigvee\{s(\alpha_1) \wedge \cdots \wedge s(\alpha_n) : \alpha_1, \ldots, \alpha_n \vdash q\}.$$

If we can recognize R from the context, then R may be dropped from $\mathrm{Con}_R(s,q)$.

Note. In any concrete application the available information for an inferential process is partial, finite and, if this is possible, not too big. In particular, this holds for the information we need to describe R. For example, it is not useful to recall that $R(x,x) = 1$ for any $x \in X$. Moreover, once it is claimed that the propositional variable x is equivalent to y at degree λ, the fact that the variable y is equivalent to x at same degree can be assumed implicitly. Then, we can admit that the similarity relation is only partially described by a relation R. In this case we have to consider the similarity generated by R and therefore, in accordance with the results in Section 3, we have

(i) to symmetrize R by considering $s(R)$,

(ii) to consider the reflexive extension $r(s(R))$ of $s(R)$,

(iii) to consider the transitive extension $\mathrm{Sim}(R) = t(r(s(R)))$ of $r(s(R))$,

(iv) to extend $\mathrm{Sim}(R)$ to the whole set of formulas by considering $\overline{\mathrm{Sim}(R)}$.

Now, it is possible to prove that if R is reflexive and symmetric, it is $\overline{\mathrm{Sim}(R)} = \mathrm{Sim}(\overline{R})$. Consequently, the steps (iii) and (iv) can be substituted by the steps

(iii′) $r(s(R))$ is extended to the whole set of formulas by considering $\overline{r(s(R))}$,

(iv′) $\overline{r(s(R))}$ becomes a similarity by considering $\mathrm{Sim}(\overline{r(s(R))})$.

5 Closure property of the consequence operator

In [2], Biacino and Gerla gave a useful characterization of the similarity-based consequence operator in terms of fuzzy closure operators. To expose such a characterization we have to define two interesting fuzzy closure operators. The first one, proposed in [1] and [5], is the operator $D : L^{P(X)} \longrightarrow L^{P(X)}$ defined by setting

$$D(s)(p) = \begin{cases} 1 & \text{if } p \text{ is a tautology,} \\ \bigvee\{s(p_1) \wedge \cdots \wedge s(p_n) : p_1, \ldots, p_n \vdash p\} & \text{otherwise.} \end{cases}$$

Such an operator is the deduction operator of a suitable fuzzy logic (the *logic of necessities*). The main properties of D are exposed in the following proposition whose proof is immediate.

Proposition 5.1. *D is a fuzzy closure operator extending the deduction operator of the classical propositional calculus. Also, for any L-subset s of formulas, $D(s)$ is* *compatible with the logical equivalence, i. e.,* $p \equiv q$ *implies* $D(s)(p) = D(s)(q)$. *Finally,* $D(s) \supseteq [s] \supseteq s$ *and* $D(s) \supseteq \mathrm{Taut}$.

We say that two fuzzy subsets of formulas s_1 and s_2 are *logically equivalent*, and we write $s_1 \equiv s_2$, provided that $D(s_1) = D(s_2)$.

The second operator we need is related with the similarity $\overline{R}$. In fact, we define $H_{\overline{R}} : L^{P(X)} \longrightarrow L^{P(X)}$ by setting, for any L-subset s of formulas,

$$H_{\overline{R}}(s) \;=\; J_{\overline{R}}(s \cup \mathrm{Taut}),$$

where $J_{\overline{R}}$ is the fuzzy closure operator associated with $\overline{R}$. It is immediate that $H_{\overline{R}}$ is a fuzzy closure operator on $L^{P(X)}$. In the sequel we will drop the index $\overline{R}$ and we write J and H to denote $J_{\overline{R}}$ and $H_{\overline{R}}$, respectively. The following theorem, whose proof was given in [2], establishes a connection among similarity logic and the logic of the necessities.

Theorem 5.1. *For any fuzzy subset s we have* $\mathrm{Con}(s, .) = (D \circ H)(s)$.

Proof. If q is a tautology, then both $\mathrm{Con}(s,q)$ and $(D \circ H)(s)(q)$ are equal to 1. Otherwise, since $\widetilde{R}(\mathrm{Taut} \cup s, B) = \bigwedge_{q \in B} H(s)(q)$, we have

$$\begin{aligned}\mathrm{Con}_R(s,q) &= \bigvee\{\widetilde{R}(\mathrm{Taut} \cup s, B) : B \vdash q \text{ and } B \text{ finite}\}\\ &= \bigvee\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_n) : \alpha_1,\dots,\alpha_n \vdash q\}\\ &= D(H(s))(q).\end{aligned}$$

□

The proof of the following theorem explains some flexibility of this characterization of Con.

Theorem 5.2. *The map* $D \circ H$ *associating to any* L*-subset* s *the* L*-subset* $\mathrm{Con}(s, .)$ *is a closure operator on* $L^{P(X)}$.

Proof. By Theorem 5.1 and Proposition 2.1 it suffices to prove that for every L-subset s of formulas, $\overline{s} = (D \circ H)(s)$ is closed with respect to H, i. e., since $\mathrm{Taut} \subseteq \overline{s}$, $J(\overline{s}) \subseteq \overline{s}$. In turn this is equivalent to prove that $\overline{s}(p) \geq \overline{s}(p') \wedge \overline{R}(p,p')$ for every $p, p' \in P(X)$. If p' is a tautology, then since $\overline{s}(p') = 1$ and $\overline{s} = (D \circ H)(s) = D(J(s \cup \mathrm{Taut})) \supseteq J(\mathrm{Taut})$, we have

$$\overline{s}(p) \geq \bigvee_{p' \in \mathrm{Taut}} \overline{R}(p,p') \geq \overline{R}(p,p') = \overline{s}(p') \wedge \overline{R}(p,p').$$

Consider the case in which p' is not a tautology. Then

$$\overline{s}(p') \;=\; \bigvee\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_m) : \alpha_1,\dots,\alpha_m \vdash p'\}.$$

Moreover, since the case $\overline{R}(p,p') = 0$ is immediate, we assume also that $\overline{R}(p,p') > 0$. Then there exists a schema $\alpha(y_1,\dots,y_n)$ such that $p = \alpha(x_1/y_1,\dots,x_n/y_n)$ and $p' = \alpha(x'_1/y_1,\dots,x'_n/y_n)$, where $(x_1,\dots,x_n)$ and $(x'_1,\dots,x'_n)$ are the related characteristics. We have that for every $\alpha_1,\dots,\alpha_m$ in $P(X)$, $\alpha_1,\dots,\alpha_m \vdash p'$ implies

$$\alpha_1,\dots,\alpha_m,\; x_1 \Rightarrow x'_1,\; x'_1 \Rightarrow x_1,\; \dots, x_n \Rightarrow x'_n,\; x'_n \Rightarrow x_n \vdash p.$$

Moreover,

$$\begin{aligned}H(s)(x_i \Rightarrow x'_i) &= J(s \cup \mathrm{Taut})(x_i \Rightarrow x'_i)\\ &\geq J(\mathrm{Taut})(x_i \Rightarrow x'_i)\\ &= \bigvee_{\alpha \in \mathrm{Taut}} \overline{R}(\alpha, x_i \Rightarrow x'_i)\\ &\geq \overline{R}(x_i \Rightarrow x_i,\, x_i \Rightarrow x'_i) = R(x_i, x'_i).\end{aligned}$$

Thus,

$$\begin{aligned}
&\overline{s}(p') \wedge \overline{R}(p,p') \\
&= \bigvee\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_m) \wedge \overline{R}(p,p') : \alpha_1,\ldots,\alpha_m \vdash p'\} \\
&\leq \bigvee\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_m) \wedge R(x_1,x_1') \wedge \cdots \wedge R(x_n,x_n') : \\
&\qquad \alpha_1,\ldots,\alpha_m,\ x_1 \Rightarrow x_1',\ x_1' \Rightarrow x_1,\ldots,\ x_n \Rightarrow x_n',\ x_n' \Rightarrow x_n \vdash p\} \\
&\leq \bigvee\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_m) \\
&\qquad\qquad \wedge H(s)(x_1 \Rightarrow x_1') \wedge H(s)(x_1' \Rightarrow x_1) \wedge \cdots \wedge H(s)(x_n' \Rightarrow x_n) : \\
&\qquad \alpha_1,\ldots,\alpha_m,\ x_1 \Rightarrow x_1',\ x_1' \Rightarrow x_1,\ \ldots,\ x_n \Rightarrow x_n',\ x_n' \Rightarrow x_n \vdash p\} \\
&\leq (D \circ H)(s)(p) = \overline{s}(p).
\end{aligned}$$

□

Notice that $H \circ D \neq D \circ H$ and therefore, in accordance with Proposition 2.2, the map $H \circ D$ is not a closure operator, in general. As an example, assume that $R(p_2, p_3) = 1$ and $R(p_1, p_j) = 0$ for every $j \neq 1$. Then, we have that

$$\begin{aligned}
H(D(\{p_3, p_2 \Rightarrow p_1\}))(p_1) &= \textstyle\bigvee_{x \in F(X)} D(\{p_3, p_2 \Rightarrow p_1\})(x) \wedge R(p_1, x) \\
&= D(\{p_3, p_2 \Rightarrow p_1\})(p_1) = 0.
\end{aligned}$$

Moreover, since p_2 and $p_2 \to p_1$ are in $H(\{p_3, p_2 \Rightarrow p_1\})$,

$$D(H(\{p_3, p_2 \Rightarrow p_1\}))(p_1) = 1.$$

We conclude this section by showing that Con is compatible with the logical equivalence of the L-subsets of formulas.

Theorem 5.3. *Let s be an L-subset of formulas. Then*

$$\mathrm{Con}(s,p) = \mathrm{Con}([s],p) = \mathrm{Con}(D(s),p).$$

Thus, for all L-subsets s_1, s_2 of formulas with $s_1 \equiv s_2$, $\mathrm{Con}(s_1, .) = \mathrm{Con}(s_2, ..)$.

Proof. It is immediate that $\mathrm{Con}(s,p) \leq \mathrm{Con}([s],p) \leq \mathrm{Con}(D(s),p)$. Also, observe that $\mathrm{Con}(s,p) = D(H(s))(p) = D(H(D(s)))(p) = \mathrm{Con}(D(s),p)$. □

6 Similarities compatible with the logical equivalence

It is worth noticing that $\overline{R}$ is not compatible with the logical equivalence. For example, let $X = \{x, y\}$, and let p be the formula $(y \Rightarrow y) \Rightarrow x$. Then, while p is logically equivalent to x, we have that $\overline{R}(x,x) = 1$ and $\overline{R}(x,p) = 0$. This suggests to substitute $\overline{R}$ with a similarity that is compatible with the logical equivalence $\equiv$.

Definition 6.1. For any p, $q \in P(X)$ the *compatible extension* of $\overline{R}$ on $P(X)$ is the L-relation $\overline{R}_{\mathrm{e}}$ defined by

$$\begin{aligned}
\overline{R}_{\mathrm{e}}(p,q) = \bigvee\{\overline{R}(p_1,p_2) \wedge \overline{R}(p_3,p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1},p_{2n}) : \\
p \equiv p_1,\ p_2 \equiv p_3,\ldots,\ p_{2n} \equiv q\}.
\end{aligned}$$

Clearly, $\overline{R}_{\mathrm{e}}$ is compatible with $\equiv$, i.e., if $p \equiv p'$ and $q \equiv q'$, then $\overline{R}_{\mathrm{e}}(p,q) = \overline{R}_{\mathrm{e}}(p',q')$, and it is easy to show that the following proposition holds.

Proposition 6.1. *If R is a similarity so is $\overline{R}_{\mathrm{e}}$. More precisely, $\overline{R}_{\mathrm{e}}$ is the smallest similarity compatible with $\equiv$ and containing $\overline{R}$.*

Proof. Immediately. □

If we set $\overline{R}_{\mathrm{e}}$ in the place of $\overline{R}$ in the definitions in the previous sections, we obtain $\widetilde{R}_{\mathrm{e}}$ instead of $\widetilde{R}$, $\mathrm{Con}_{\mathrm{e}}(s,q)$ instead of $\mathrm{Con}(s,q)$, J_{e} instead of J, and H_{e} instead of H.

As an example, for any $s \in L^{P(X)}$ and $B \subseteq P(X)$, we set

$$\widetilde{R}_e(s, B) = \bigwedge_{q\in B} \bigvee_{p\in P(X)} (s(p) \wedge \overline{R}_e(p, q)).$$

It is easy to prove that for any $A, B \subseteq P(X)$, $B \subseteq [A]$ implies $\widetilde{R}_e(A, B) = 1$. Several properties we have proved for $\widetilde{R}$, $\mathrm{Con}(s, q)$, J and H are valid also for $\widetilde{R}_e$, $\mathrm{Con}_e(s, q)$, J_e and H_e, respectively. As an example, it is immediate to prove that for any $s \in L^{P(X)}$ and $A, B \subseteq P(X)$ we have $\widetilde{R}_e(s, A) \wedge \widetilde{R}_e(A, B) \leq \widetilde{R}_e(s, B)$. Also, $\mathrm{Con}_e(s, .) = (D \circ H_e)(s)$.

Lemma 6.1. *Let s be an L-subset of formulas containing* Taut*, and assume that s is closed with respect to $\equiv$ and $\overline{R}$. Then, for any $q \in P(X)$,* $\mathrm{Con}(s, q) = \mathrm{Con}_e(s, q)$.

Proof. Clearly, $\mathrm{Con}(s, q) \leq \mathrm{Con}_e(s, q)$. Conversely, observe that if $B \subseteq P(X)$,

$$\begin{aligned}
&\widetilde{R}_e(\mathrm{Taut} \cup s, B) = \widetilde{R}_e(s, B)\\
&= \bigwedge_{r\in B} \bigvee_{p\in P(X)} (s(p) \wedge \overline{R}_e(p, r))\\
&= \bigwedge_{r\in B} \bigvee_{p\in P(X)} (s(p) \wedge \bigvee\{\overline{R}(p_1, p_2) \wedge \overline{R}(p_3, p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1}, p_{2n}) :\\
&\qquad\qquad p_1 \equiv p,\ p_2 \equiv p_3,\ \ldots,\ p_{2n} \equiv r\})\\
&= \bigwedge_{r\in B} \bigvee_{p\in P(X)} \bigvee\{s(p) \wedge \overline{R}(p_1, p_2) \wedge \overline{R}(p_3, p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1}, p_{2n}) :\\
&\qquad\qquad p_1 \equiv p,\ p_2 \equiv p_3,\ \ldots,\ p_{2n} \equiv r\}.
\end{aligned}$$

Since s is closed with respect to $\equiv$ and $\overline{R}$, if $p_1 \equiv p,\ p_2 \equiv p_3,\ \ldots,\ p_{2n} \equiv r$, then we have that

$$\begin{aligned}
&s(p) \wedge \overline{R}(p_1, p_2) \wedge \overline{R}(p_3, p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1}, p_{2n})\\
&\quad = s(p_1) \wedge \overline{R}(p_1, p_2) \wedge \overline{R}(p_3, p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1}, p_{2n})\\
&\quad \leq s(p_2) \wedge \overline{R}(p_3, p_4) \wedge \cdots \wedge \overline{R}(p_{2n-1}, p_{2n})\\
&\qquad \cdots\\
&\quad \leq s(p_{2n-1}) \wedge \overline{R}(p_{2n-1}, p_{2n}),
\end{aligned}$$

whence, we obtain

$$\widetilde{R}_e(s, B) \leq \bigwedge_{r\in[B]} \bigvee_{p\in P(X)} (s(p) \wedge \overline{R}(p, r) = \widetilde{R}(s, [B])).$$

Thus, since $B \vdash q$ implies $[B] \vdash q$, we can conclude that

$$\begin{aligned}
\mathrm{Con}_e(s, q) &= \bigvee\{\widetilde{R}_e(\mathrm{Taut} \cup s, B) : B \vdash q\}\\
&\leq \bigvee\{\widetilde{R}(s, [B]) : B \vdash q\}\\
&\leq \bigvee\{\widetilde{R}(s, D) : D \vdash q\} = \mathrm{Con}(s, q).
\end{aligned}$$

□

The following theorem shows that the consequence relation based on $\overline{R}_e$ coincides with the one based on $\overline{R}$.

Theorem 6.1. *The consequence relations* Con *and* Con_e *coincide.*

Proof. First we observe that, by Proposition 5.1, $D(H(s))$ contains Taut and it is closed with respect to $\equiv$. Moreover, since $D \circ H$ is a closure operator, by Proposition 2.1 we have that $D(H(s)) = H(D(H(s)) = J(D(H(s))$ and therefore $D(H(s))$ is a fixed point of J, i.e. $D(H(s))$ is closed with respect to $\overline{R}$, too. Thus, by the just proven lemma,

$$\begin{aligned}
\mathrm{Con}(s, .) &= D(H((s)) = D(H(D(H((s))))\\
&= D(H_e(D(H(s)))) \supseteq D(H_e(s)) = \mathrm{Con}_e(s, .).
\end{aligned}$$

□

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